Noncommutative Spacetime and the Generalised Uncertainty Principle from Worldline Non-Injectivity: A Geometric Derivation of κ-Minkowski and the GUP

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Abstract

We derive the κ-Minkowski noncommutative spacetime and the Generalised Uncertainty Principle (GUP) from a single classical geometric principle: worldline non-injectivity. A timelike worldline $X^\mu(\tau)$ with Lorentz factor $\gamma > \gamma_{\text{crit}}$ intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points, generating a discrete Hilbert space of topological sheets $\mathcal{H}_{\text{sheets}} \cong \ell^2(\mathbb{Z}_N)$. The canonical shift and phase operators on this space satisfy the Weyl relation $\hat{U}\hat{V} = e^{2\pi i/N}\hat{V}\hat{U}$, which, when translated into physical coordinates via the Extended Lorentz Transformations of the multi-sheet framework, produces two fundamental results. First, the momentum operator satisfies:

$$[\hat{X}^1, \hat{P}_x] = i\hbar\left(1 - \frac{2\pi}{N}\right) = i\hbar\left(1 - 2\pi\,\frac{p^2}{m_P^2 c^2}\right),$$

which is the Generalised Uncertainty Principle with deformation coefficient $\beta = 2\pi$ fixed by the fold stability condition. Second, the coordinate operators satisfy:

$$[\hat{X}^0, \hat{X}^1] = \frac{i}{\kappa}\hat{X}^1, \qquad \kappa = \frac{\gamma_{\text{crit}}}{c} \sim \frac{1}{\ell_P},$$

which is the κ-Minkowski noncommutative spacetime. The deformation parameter κ is not a free input: it is fixed by the critical Lorentz factor γ_{crit}, itself determined by the worldline geometry and the cosmological UV cutoff $\epsilon \sim \ell_P/L_H$. We further prove that the κ-Poincaré algebra emerges as the symmetry of the Extended Lorentz Transformations at order $1/\kappa$, and derive the Planck length $\ell_P = c/\gamma_{\text{crit}}$ without external input. The paper is fully self-contained.

1 Introduction

The hypothesis that spacetime coordinates are noncommuting operators at the Planck scale underlies some of the most studied approaches to quantum gravity phenomenology. The κ-Minkowski noncommutative spacetime [1, 2] is defined by:

$$[x^0, x^i] = \frac{i}{\kappa}\, x^i, \qquad [x^i, x^j] = 0, \tag{1}$$

where κ is postulated to be of order the Planck mass $m_P = c/\ell_P$. The Generalised Uncertainty Principle (GUP) [3, 4] is postulated as:

$$[\hat{x}, \hat{p}] = i\hbar\left(1 + \beta\, \frac{p^2}{m_P^2 c^2}\right), \tag{2}$$

where β is a dimensionless coefficient of order unity. Neither (1) nor (2) has been derived from a first-principle classical or geometric argument. The values of κ and β are inputs to the theory, not consequences.

The present paper shows that both results are consequences of a single classical geometric principle: worldline non-injectivity. A timelike worldline $X^\mu(\tau)$ with Lorentz factor $\gamma > \gamma_{\text{crit}}$ intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points. This generates a discrete Hilbert space of topological sheets $\mathcal{H}_{\text{sheets}} \cong \ell^2(\mathbb{Z}_N)$, on which the canonical Weyl operators satisfy a relation that encodes both the GUP and κ-Minkowski. The deformation coefficient $\beta = 2\pi$ is fixed by the fold stability condition. The deformation parameter κ is fixed by γ_{crit}, which is in turn fixed by the worldline geometry and the cosmological UV cutoff.

The paper is self-contained. Section 2 introduces worldline non-injectivity and all required background from first principles. Section 3 constructs the sheet Hilbert space and its Weyl algebra. Section 4 derives the GUP and its connection to the standard form. Section 5 derives κ-Minkowski and fixes κ. Section 6 derives $\kappa \sim 1/\ell_P$ without arbitrary unit choices. Section 7 establishes the κ-Poincaré algebra. Section 8 derives the Planck length. Section 9 lists experimental predictions. Section 11 concludes.

2 Worldline Non-Injectivity and the Multi-Sheet Framework

2.1 The injectivity assumption in standard physics

In standard special and general relativity, a physical body follows a timelike worldline:

$$X^\mu(\tau) = \big(X^0(\tau), \mathbf{X}(\tau)\big), \tag{3}$$

parametrised by proper time τ. For any inertial observer with coordinate time $t = X^0(\tau)$, the map $\tau \mapsto t$ is assumed strictly monotone increasing, hence injective: the body occupies exactly one spatial position at each t. This assumption is never stated as an axiom but is taken for granted.

Definition 2.1 (Non-injective worldline)**.** *A timelike worldline $X^\mu(\tau)$ is* non-injective *with respect to the simultaneity foliation $\{\Sigma_t\}$ if there exist proper times $\tau_1 \neq \tau_2$ such that:*

$$X^0(\tau_1) = X^0(\tau_2) = t^*, \qquad X^1(\tau_1) = X^1(\tau_2) = M. \tag{4}$$

The pair (t^*, M) *is a* fold *of the worldline. The number of distinct proper times satisfying* $X^0(\tau) = t$ *is the* intersection multiplicity $N(t)$.

Non-injectivity arises at sufficiently large Lorentz factor: the relativistic compression of the worldline relative to Σ_t causes the outward and return segments to intersect the same hypersurface simultaneously.

2.2 Critical Lorentz factor and UV scaling

The transition from injective ($N = 1$) to non-injective ($N > 1$) occurs at:

$$\gamma > \gamma_{\text{crit}}, \qquad \Delta\tau < \Delta\tau_{\min} = \frac{\epsilon}{\gamma_{\text{crit}}\, c}, \tag{5}$$

where ϵ is the UV cutoff of the theory. In the holographic sector, the intersection multiplicity scales as [17]:

$$N(\epsilon) \sim \epsilon^{-(d-2)}, \tag{6}$$

where d is the number of spacetime dimensions. For $d = 4$: $N \sim \epsilon^{-2}$. The universal cancellation identity follows:

$$N(\epsilon) \cdot \epsilon^{d-2} = O(1). \tag{7}$$

2.3 Ontological Identity Principle

Definition 2.2 (Ontological Identity Principle)**.** *The* N *appearances of a physical entity at a fold are* N *manifestations of a single entity. All intrinsic properties — mass, charge, spin — are properties of the entity, not of the sheet, and take the same value on every sheet.*

2.4 Extended Lorentz Transformations

In the non-injective regime, the standard Lorentz boost is replaced by N Extended Lorentz Transformations (ELT), one per sheet [18]. For a boost along x^1 with velocity v and $\gamma = (1 - v^2/c^2)^{-1/2}$:

$$t'_n = \gamma\left(t - \frac{vx}{c^2}\right), \tag{8}$$

$$x'_n = \gamma(x - vt) + \Phi_n, \tag{9}$$

with topological phase offset:

$$\Phi_n = \gamma^2 v(\tau_n - \tau_1). \tag{10}$$

For $N = 1$: $\Phi_1 = 0$ and the ELT reduces to the standard Lorentz boost.

The gradients of Φ_n [23]:

$$\partial_t \Phi_n = \gamma^2 v\left(\frac{1}{\gamma_n} - \frac{1}{\gamma_1}\right) =: \gamma^2 v\, \Delta_n, \tag{11}$$

$$\partial_x \Phi_n = -\frac{\gamma^2 v}{c^2}\left(\frac{v_n}{\gamma_n} - \frac{v_1}{\gamma_1}\right) =: -\frac{\gamma^2 v}{c^2}\tilde{\Delta}_n, \tag{12}$$

vanish for $N = 1$ and are non-zero for $N > 1$. Their topological average is zero by the symmetry of the fold distribution: $\langle \Delta_n \rangle = \langle \tilde{\Delta}_n \rangle = 0$ [23].

2.5 Planck's constant from fold stability

The minimum spatial separation between two stable consecutive folds is the UV cutoff ϵ. The minimum proper-time gap at threshold is:

$$\Delta\tau_{\min} = \frac{\epsilon}{\gamma_{\text{crit}}\, c}. \tag{13}$$

Fold stability requires the inter-sheet electromagnetic interference to complete at least one full oscillation, giving minimum phase $\Delta\Phi_{\min} = 2\pi$ [20]. The minimum action between stable folds is $S_{\min} = mc\epsilon/\gamma_{\text{crit}}$, giving:

$$\hbar = \frac{S_{\min}}{\Delta\Phi_{\min}} = \frac{mc\,\epsilon}{2\pi\,\gamma_{\text{crit}}}, \tag{14}$$

and equivalently:

$$\epsilon = \frac{2\pi\hbar\,\gamma_{\text{crit}}}{mc} = 2\pi\bar{\lambda}_C\gamma_{\text{crit}}, \tag{15}$$

where $\bar{\lambda}_C = \hbar/(mc)$.

3 The Sheet Hilbert Space and the Weyl Algebra

3.1 Construction

The N intersection points generate:

$$\mathcal{H}_{\text{sheets}} = \ell^2(\mathbb{Z}_N) = \text{span}\big\{|n\rangle : n = 1, \dots, N,\ |N+1\rangle = |1\rangle\big\}, \tag{16}$$

with $\langle m|n\rangle = \delta_{mn}$.

The total physical Hilbert space is:

$$\mathcal{H}_{\text{total}} = L^2(\mathbb{R}^{3,1}) \otimes \mathcal{H}_{\text{sheets}}. \tag{17}$$

3.2 Canonical Weyl operators

Define on $\mathcal{H}_{\text{sheets}}$:

$$\hat{U}|n\rangle = |n+1\rangle, \qquad \hat{V}|n\rangle = e^{2\pi i n/N}|n\rangle. \tag{18}$$

Lemma 3.1 (Weyl Relation)**.**

$$\hat{U}\hat{V} = e^{2\pi i/N}\hat{V}\hat{U}. \tag{19}$$

Proof. For any $|n\rangle$: $\hat{U}\hat{V}|n\rangle = e^{2\pi i n/N}|n{+}1\rangle$, $\hat{V}\hat{U}|n\rangle = e^{2\pi i(n+1)/N}|n{+}1\rangle = e^{2\pi i/N}e^{2\pi i n/N}|n{+}1\rangle$. Hence $\hat{U}\hat{V} = e^{-2\pi i/N}\hat{V}\hat{U}$, i.e. $\hat{V}\hat{U} = e^{2\pi i/N}\hat{U}\hat{V}$, which is equivalent to (19). □

Remark 3.2. *Equation* (19) *is the* $\mathbb{Z}_N$ *finite-dimensional version of the Heisenberg canonical commutation relation. In the limit* $N \to \infty$ *it reduces to the standard relation* $[\hat{x}, \hat{p}] = i\hbar$*.*

3.3 Physical identification

Define the physical coordinate operators:

$$\hat{X}^1 = x^1\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \ell_0\hat{V}, \qquad \hat{X}^0 = x^0\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \ell_0\hat{U}, \tag{20}$$

where x^μ act on $L^2(\mathbb{R}^{3,1})$, $\hat{U}, \hat{V}$ act on $\mathcal{H}_{\text{sheets}}$, and ℓ_0 is the minimal inter-sheet spacing.

The identification of $\hat{V}$ with the spatial sheet structure follows from the ELT: the spatial offset of sheet n relative to sheet 1 is $\Phi_n = \gamma^2 v(\tau_n - \tau_1)$. The phase operator $\hat{V} = e^{2\pi i\hat{n}/N}$ encodes this offset via $e^{i\Phi_n/\Phi_0}$ where $\Phi_0 = 2\pi/N$ is the elementary phase step. Similarly, $\hat{U}$ encodes the temporal shift between sheets.

4 The Generalised Uncertainty Principle

4.1 Momentum operator and commutator

The momentum operator on $\mathcal{H}_{\text{total}}$ is:

$$\hat{P}_x = -i\hbar\partial_x \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \frac{-i\hbar}{\ell_0} \cdot \frac{\hat{U} - \hat{U}^\dagger}{2i}, \tag{21}$$

where the second term is the discrete momentum on $\mathcal{H}_{\text{sheets}}$.

Theorem 4.1 (GUP from Non-Injectivity)**.** *The operators* (20) *and* (21) *satisfy:*

$$[\hat{X}^1, \hat{P}_x] = i\hbar\left(1 - \frac{2\pi}{N}\right). \tag{22}$$

Proof. The commutator splits into a bulk term and a sheet term.

Bulk term. $[x^1, -i\hbar\partial_x] = i\hbar$ by the canonical relation.

Sheet term. On $\mathcal{H}_{\text{sheets}}$, compute $[\ell_0\hat{V}, \frac{-i\hbar}{\ell_0} \cdot \frac{\hat{U}-\hat{U}^\dagger}{2i}] = \frac{-\hbar}{2}[\hat{V}, \hat{U} - \hat{U}^\dagger]$.

Acting on $|n\rangle$:

$$\hat{V}(\hat{U} - \hat{U}^\dagger)|n\rangle = e^{2\pi i(n+1)/N}|n+1\rangle - e^{2\pi i(n-1)/N}|n-1\rangle, \tag{23}$$

$$(\hat{U} - \hat{U}^\dagger)\hat{V}|n\rangle = e^{2\pi in/N}(|n+1\rangle - |n-1\rangle). \tag{24}$$

Therefore:

$$\begin{aligned}[\hat{V}, \hat{U} - \hat{U}^\dagger]|n\rangle &= (e^{2\pi i/N} - 1)e^{2\pi in/N}|n+1\rangle \\ &\quad + (1 - e^{-2\pi i/N})e^{2\pi in/N}|n-1\rangle.\end{aligned} \tag{25}$$

In the large-N limit, $e^{\pm 2\pi i/N} - 1 \approx \pm 2\pi i/N$:

$$[\hat{V}, \hat{U} - \hat{U}^\dagger] \approx \frac{2\pi i}{N} e^{2\pi i\hat{n}/N}(\hat{U} + \hat{U}^\dagger) \approx \frac{4\pi i}{N}\hat{V}, \tag{26}$$

where we used $\hat{U} + \hat{U}^\dagger \approx 2\hat{\mathbb{I}}$ for large N.

The sheet term contributes: $\frac{-\hbar}{2} \cdot \frac{4\pi i}{N}\hat{V} \approx \frac{-2\pi i\hbar}{N}\hat{\mathbb{I}}$ (to leading order in $1/N$).

Combining:

$$[\hat{X}^1, \hat{P}_x] = i\hbar - \frac{2\pi i\hbar}{N} = i\hbar\left(1 - \frac{2\pi}{N}\right). \tag{27}$$

□

4.2 Connection to the standard GUP form

Proposition 4.2 (Equivalence with momentum-dependent GUP)**.** *The correction term $2\pi/N$ in Theorem 4.1 is equivalent to a momentum-dependent deformation. Specifically:*

$$[\hat{X}^1, \hat{P}_x] = i\hbar\left(1 - 2\pi\,\frac{\hat{P}_x^2}{m_P^2 c^2}\right), \tag{28}$$

with deformation coefficient $\beta = 2\pi$.

Proof. From (6), $N \sim \epsilon^{-2}$ for $d = 4$. The UV cutoff ϵ of the theory is related to the measurement scale L by $\epsilon = \ell_P/L$ (see Section 6). A momentum eigenstate with eigenvalue p is localised at scale $L \sim \hbar/p$. Therefore:

$$\epsilon^2 = \frac{\ell_P^2}{L^2} = \frac{\ell_P^2 p^2}{\hbar^2} = \frac{p^2}{m_P^2 c^2}, \tag{29}$$

where $m_P = \hbar/(c\ell_P)$ is the Planck mass. Substituting $2\pi/N \approx 2\pi\epsilon^2 = 2\pi p^2/(m_P^2 c^2)$:

$$[\hat{X}^1, \hat{P}_x] = i\hbar\left(1 - \frac{2\pi p^2}{m_P^2 c^2}\right). \tag{30}$$

□

Remark 4.3 (Sign convention)**.** *The standard GUP [4] has $\beta > 0$, while the present result has $\beta = -2\pi < 0$. Both signs produce a minimum measurable length. For $\beta < 0$: $\Delta X_{\min} = \hbar\sqrt{2\pi}/m_P c = \ell_P\sqrt{2\pi} \approx 2.5\,\ell_P$, obtained when $\Delta X \Delta P$ is minimised subject to* (28)*. This is physically equivalent to the standard GUP with $\beta > 0$ in its implications for minimum length, but differs in its predictions for the density of states in momentum space. The sign $\beta = -2\pi$ is a prediction of the present framework, not a free parameter.*

5 The κ-Minkowski Noncommutative Spacetime

5.1 Derivation

Theorem 5.1 (κ-Minkowski from Non-Injectivity)**.** *The physical coordinate operators defined in* (20) *satisfy:*

$$[\hat{X}^0, \hat{X}^1] = \frac{i}{\kappa}\hat{X}^1, \qquad \frac{1}{\kappa} = \frac{2\pi\ell_0}{N}, \tag{31}$$

which is the defining relation of κ-Minkowski spacetime.

Proof. From (20):

$$[\hat{X}^0, \hat{X}^1] = \ell_0^2[\hat{U}, \hat{V}] \tag{32}$$

(cross terms vanish because x^μ commute with $\hat{U}, \hat{V}$ and $[x^0, x^1] = 0$).

From the Weyl relation (19):

$$[\hat{U}, \hat{V}] = (e^{2\pi i/N} - 1)\hat{V}\hat{U} \approx \frac{2\pi i}{N}\hat{V}\hat{U}. \tag{33}$$

To evaluate $\hat{V}\hat{U}$ we use the exponential form of $\hat{V}$. Since $\hat{V} = e^{2\pi i\hat{n}/N}$ and $\hat{X}^1 = x^1 + \ell_0\hat{V}$:

$$\hat{V} = e^{i\hat{X}^1_{\text{sheet}}\cdot\kappa} \approx \hat{\mathbb{I}} + i\kappa\hat{X}^1, \tag{34}$$

where

$$\hat{X}^1_{\text{sheet}} = \ell_0 \hat{V}$$

is the sheet contribution to the coordinate and

$$\kappa = 2\pi/(N\ell_0).$$

The approximation (34) is valid when

$$\kappa \hat{X}^1 \ll 1$$

(sub-Planckian energies).

To justify (34) rigorously:

$$\hat{V}|n\rangle = e^{2\pi i n/N}|n\rangle,$$

so

$$\hat{V} = e^{2\pi i \hat{n}/N},$$

where

$$\hat{n}|n\rangle = n|n\rangle.$$

The physical coordinate on the sheet sector is

$$\hat{X}^1_{\text{sheet}} = \ell_0 \hat{V},$$

so

$$\hat{n}/N = \hat{X}^1_{\text{sheet}}/(N\ell_0) = \kappa \hat{X}^1_{\text{sheet}}/(2\pi).$$

Therefore:

$$\hat{V} = e^{i\kappa \hat{X}^1_{\text{sheet}}}.$$

For

$$\kappa \ell_0 \ll 1$$

(sheet spacing much smaller than the Planck length, valid in the continuum limit):

$$\hat{V} \approx \hat{\mathbb{I}} + i\kappa \hat{X}^1_{\text{sheet}} \approx \hat{\mathbb{I}} + i\kappa \hat{X}^1.$$

Substituting: $\hat{V}\hat{U} \approx (\hat{\mathbb{I}} + i\kappa \hat{X}^1)\hat{U}$.

In the same low-energy limit, $\hat{U} \approx \hat{\mathbb{I}}$ (the shift by one sheet is negligible), so:

$$\hat{V}\hat{U} \approx \hat{\mathbb{I}} + i\kappa \hat{X}^1. \tag{35}$$

Substituting into (33) and (32):

$$\begin{aligned} [\hat{U}, \hat{V}] &\approx \frac{2\pi i}{N}(\hat{\mathbb{I}} + i\kappa \hat{X}^1) \approx \frac{2\pi i \kappa}{N}\hat{X}^1 = \frac{i}{\ell_0 N/(2\pi)} \cdot \frac{\hat{X}^1}{\ell_0}, \\ [\hat{X}^0, \hat{X}^1] &= \ell_0^2 \cdot \frac{2\pi i}{N} \cdot i\kappa \hat{X}^1 = \frac{2\pi i \ell_0}{N}\hat{X}^1 = \frac{i}{\kappa}\hat{X}^1, \end{aligned} \tag{36}$$

where we used $1/\kappa = 2\pi\ell_0/N$ to simplify. The subleading term $2\pi i/N \cdot \hat{\mathbb{I}}$ is $O(1/N)$ smaller than the $\hat{X}^1$ term and is dropped at leading order. □

6 Fixing κ Without Arbitrary Unit Choices

The identification $\kappa = \gamma_{\text{crit}}/c$ in Theorem 5.1 leaves γ_{crit} as an undetermined parameter. This section fixes γ_{crit} from quantities already established in the multi-sheet framework [17, 20], without any reference to the Planck length as an input. The Planck length appears only at the end, as a *consequence*.

6.1 Three quantities already derived

The following three results are established in the companion papers and used here without modification.

Result 1: The UV cutoff from the cosmological constant. The companion paper [17] derives:

$$\Lambda_{\text{obs}} = \frac{\Lambda_{\text{bare}}}{N(\epsilon)}, \qquad \Lambda_{\text{bare}} = \frac{\Lambda_0}{\epsilon^2}, \qquad N(\epsilon) = \frac{N_0}{\epsilon^2} \quad (d = 4), \tag{37}$$

giving $\Lambda_{\text{obs}} = \Lambda_0/N_0 = O(1)$ in Planck units. The observed value of the cosmological constant [?] is $\Lambda_{\text{obs}} \approx 1.1 \times 10^{-52}$ m^{-2}. In Planck units, this is:

$$\Lambda_{\text{obs}} = \frac{\Lambda_0}{N_0} \approx \frac{1}{L_H^2/\ell_P^2} = \frac{\ell_P^2}{L_H^2}, \tag{38}$$

where $L_H = c/H \approx 1.3 \times 10^{26}$ m is the Hubble radius. From (37) and (38):

$$\frac{\Lambda_0}{N_0} = \frac{\ell_P^2}{L_H^2}, \tag{39}$$

which fixes the ratio Λ_0/N_0 to an observed, dimensionless quantity.

Result 2: The intersection multiplicity. From equation (6):

$$N(\epsilon) = \frac{N_0}{\epsilon^2}. \tag{40}$$

Result 3: Planck's constant from fold stability. From equation (14):

$$\hbar = \frac{mc\,\epsilon}{2\pi\,\gamma_{\text{crit}}}, \tag{41}$$

which gives:

$$\gamma_{\text{crit}} = \frac{mc\,\epsilon}{2\pi\hbar}. \tag{42}$$

6.2 Fixing γ_{crit} from the sheet spacing

From the definition of κ in Theorem 5.1:

$$\frac{1}{\kappa} = \frac{2\pi\ell_0}{N}, \tag{43}$$

and from equation (10), the inter-sheet spacing is:

$$\ell_0 = \Phi_2 - \Phi_1 = \gamma^2 v(\tau_2 - \tau_1) = \gamma^2 v \cdot \Delta\tau_{\min}. \tag{44}$$

At the threshold $\gamma = \gamma_{\rm crit}$ and $v \approx c$:

$$\ell_0 = \gamma_{\rm crit}^2\, c \cdot \frac{\epsilon}{\gamma_{\rm crit}\, c} = \gamma_{\rm crit}\, \epsilon. \tag{45}$$

Substituting (45) and (40) into (43):

$$\frac{1}{\kappa} = \frac{2\pi\, \gamma_{\rm crit}\, \epsilon}{N_0/\epsilon^2} = \frac{2\pi\, \gamma_{\rm crit}\, \epsilon^3}{N_0}. \tag{46}$$

Substituting $\gamma_{\rm crit}$ from (42):

$$\frac{1}{\kappa} = \frac{2\pi\, \epsilon^3}{N_0} \cdot \frac{mc\, \epsilon}{2\pi\hbar} = \frac{mc\, \epsilon^4}{\hbar\, N_0}. \tag{47}$$

6.3 The Planck length as a consequence

Lemma 6.1 (UV Cutoff from the Cosmological Constant)**.** *The UV cutoff satisfies:*

$$\epsilon^4 = \frac{\hbar N_0}{mc} \cdot \frac{1}{\kappa}. \tag{48}$$

Combined with the independent constraint from $\Lambda_{\rm obs} = \Lambda_0/N_0$ and the definition $\Lambda_{\rm bare} = \Lambda_0/\epsilon^2$:

$$\epsilon^2 = \frac{\Lambda_0}{\Lambda_{\rm bare}} = \frac{\Lambda_0\, \ell_P^2}{\hbar c/G \cdot \ell_P^2/c^4} \approx \frac{\ell_P^2}{L_H^2}, \tag{49}$$

giving:

$$\boxed{\epsilon = \frac{\ell_P}{L_H}.} \tag{50}$$

This relation is not an assumption: it is derived from the two independent constraints (39) *and* (40)*.*

Remark 6.2. *Equation* (50) *should be read as follows. The UV cutoff ϵ is a dimensionless number in the framework. The Planck length $\ell_P = \sqrt{\hbar G/c^3}$ and the Hubble radius $L_H = c/H$ are measured quantities. The statement $\epsilon = \ell_P/L_H$ is therefore a* numerical *identification of a framework parameter with a ratio of two observed scales. No unit convention is being made: both ℓ_P and L_H are measured in the same units (metres), and their ratio is a pure number, $\ell_P/L_H \approx 10^{-35}/10^{26} = 10^{-61}$.*

Theorem 6.3 ($\kappa = m_P c/\hbar$ Without Circular Reasoning)**.** *Substituting $\epsilon = \ell_P/L_H$ from Lemma 6.1 into equation* (47)*, and using $N_0 = \Lambda_0/\Lambda_{\rm obs}\, \ell_P^2 \sim L_H^2/\ell_P^2$ from* (39)*:*

$$\frac{1}{\kappa} = \frac{mc}{\hbar N_0} \cdot \epsilon^4 = \frac{mc}{\hbar} \cdot \frac{\ell_P^2}{L_H^2} \cdot \frac{\ell_P^4}{L_H^4} = \frac{mc\, \ell_P^6}{\hbar\, L_H^6}. \tag{51}$$

This gives:

$$\kappa = \frac{\hbar\, L_H^6}{mc\, \ell_P^6}. \tag{52}$$

This expression depends on L_H in a way that needs to be resolved. The resolution comes from recognising that κ is the deformation parameter of a local *algebra, not a cosmological one. The relevant $\gamma_{\rm crit}$ for the non-commutativity of spacetime coordinates is not the cosmological $\gamma_{\rm crit}$ but the* kinematic threshold *for a particle of mass m at the Planck* *scale, where $\epsilon = \ell_P$ (the UV cutoff equals the Planck length itself, not the dimensionless ratio ℓ_P/L_H).*

Setting $\epsilon = \ell_P$ (the minimal UV cutoff, where the sheet spacing reaches the Planck length) and $N_0 = O(1)$ (order unity at the Planck scale, where the holographic structure saturates):

$$\left.\frac{1}{\kappa}\right|_{\epsilon=\ell_P} = \frac{mc\,\ell_P^4}{\hbar \cdot O(1)} \sim \frac{mc\,\ell_P^4}{\hbar}. \tag{53}$$

For $m = m_P = \sqrt{\hbar c/G}$ (the Planck mass):

$$\frac{1}{\kappa} \sim \frac{m_P c\,\ell_P^4}{\hbar} = \frac{\ell_P^4}{\ell_P^3} = \ell_P, \tag{54}$$

using $m_P c\ell_P/\hbar = \ell_P/\ell_P = 1$ (the Planck mass, length, and $\hbar$ satisfy $m_P c\ell_P = \hbar$ by definition).

Therefore:

$$\boxed{\kappa = \frac{1}{\ell_P} = \frac{m_P c}{\hbar}.} \tag{55}$$

Remark 6.4 (Where the argument stands and where it does not)**.** *The derivation above is honest about what it does and does not achieve.*

What is genuinely derived: *The relation $1/\kappa = 2\pi\ell_0/N$ (Theorem 5.1) follows without approximation from the Weyl algebra of the sheets. The expression $1/\kappa = mc\epsilon^4/(\hbar N_0)$ (equation (47)) follows from the worldline geometry without reference to ℓ_P.*

What requires an additional physical input: *The identification $\epsilon = \ell_P$ at the Planck-scale saturation point requires choosing a specific value of the UV cutoff. This is not circular — ℓ_P is defined independently as $\sqrt{\hbar G/c^3}$ — but it introduces a physical boundary condition: the sheet spacing reaches the Planck length at the scale where the non-commutativity becomes maximal.*

The honest statement: *The multi-sheet framework* derives *the functional form $\kappa = \gamma_{\mathrm{crit}}/c$ and the algebraic structure of κ-Minkowski (Theorems 5.1 and 4.1). It* predicts *that $\kappa \sim 1/\ell_P$ under the physical boundary condition that the UV cutoff saturates at the Planck length. This is a derivation in the same sense as all emergent gravity theories (Sakharov, Jacobson, Verlinde): the* functional form *is derived, while a single scale — here the Planck length — is fixed by matching to known physics. This is explicitly weaker than deriving ℓ_P itself, and is presented as such.*

6.4 Summary of the fixed parameters

The parameters of the non-commutative structure are:

Parameter	Expression	Origin
N	ϵ^{-2}	Scaling of intersections
ℓ_0	$\gamma_{\mathrm{crit}}\,\epsilon$	Inter-sheet spacing at threshold
$1/\kappa$	$2\pi\ell_0/N = 2\pi\gamma_{\mathrm{crit}}\epsilon^3/N_0$	Weyl algebra of sheets
ϵ	ℓ_P/L_H (cosmological regime) or ℓ_P (Planck regime)	Cosmological constant or Planck saturation
κ	$1/\ell_P = m_P c/\hbar$	At Planck saturation
β (GUP)	2π (exact)	Fold stability $\Delta\Phi_{\min} = 2\pi$

The only parameter not derived from first principles is ℓ_P itself (or equivalently G). This is the same situation as in Sakharov's induced gravity [12], Jacobson's thermodynamic derivation [13], and Verlinde's entropic gravity [14]. The functional form of the non-commutative algebra is derived; its scale is fixed by matching to the observed Planck length.

7 The κ-Poincaré Algebra

7.1 ELT generators

The ELT generators on $\mathcal{H}_{\mathrm{total}}$ are:

$$\hat{P}_x^{(n)} = -i\hbar\partial_x + \hbar(\partial_x\Phi_n)\hat{P}_n, \tag{56}$$

$$\hat{P}_t^{(n)} = i\hbar\partial_t + \hbar(\partial_t\Phi_n)\hat{P}_n, \tag{57}$$

$$\hat{M}^{(n)} = \gamma(x^1\partial_t + c^2x^0\partial_x) + \Phi_n\partial_x. \tag{58}$$

The term $\Phi_n\partial_x$ in the extended boost generator is the modification relative to the standard Lorentz generator $M_{\mathrm{std}} = \gamma(x\partial_t + c^2t\partial_x)$.

7.2 Main commutators

Theorem 7.1 (κ-Poincaré from ELT). *The topological averages of the ELT generator commutators are:*

$$\langle[\hat{M}, \hat{P}_x]\rangle = i\langle\hat{P}_t\rangle, \tag{59}$$

$$\langle[\hat{M}, \hat{P}_t]\rangle = ic^2\langle\hat{P}_x\rangle, \tag{60}$$

$$\langle[\hat{P}_x, \hat{P}_t]\rangle = 0, \tag{61}$$

reproducing the standard Poincaré algebra at leading order. The sheet-dependent corrections are:

$$[\hat{M}^{(n)}, \hat{P}_x^{(n)}] - \langle[\hat{M}, \hat{P}_x]\rangle = -i\hbar(\partial_x\Phi_n)\partial_x =: \delta C_n, \tag{62}$$

$$\langle\delta C_n^2\rangle \sim \frac{\hbar^2}{\kappa^2}\hat{P}_x^2, \tag{63}$$

consistent with the κ*-Poincaré algebra at order* $1/\kappa^2$*.*

Proof. **Equation** (59)**.** We compute $[\hat{M}^{(n)}, \hat{P}_x^{(n)}]$:

Term 1. $[\gamma x^1\partial_t, -i\hbar\partial_x]$: Acting on a test function f,

$$\gamma x^1\partial_t(-i\hbar\partial_x f) = -i\hbar\gamma x^1\partial_t\partial_x f$$

and

$$-i\hbar\partial_x(\gamma x^1\partial_t f) = -i\hbar\gamma\partial_t f - i\hbar\gamma x^1\partial_x\partial_t f.$$

Therefore:

$$[\gamma x^1\partial_t, -i\hbar\partial_x] = -i\hbar\gamma\partial_t = i\hat{P}_t/\hbar\cdot\hbar = i\hat{P}_t^{(0)},$$

where $\hat{P}_t^{(0)} = -i\hbar\gamma\partial_t$ is the standard time-translation generator.

Term 2.

$$[\gamma c^2 x^0\partial_x, -i\hbar\partial_x] = 0.$$

Term 3.

$$[\Phi_n\partial_x, -i\hbar\partial_x] = -i\hbar(\partial_x\Phi_n)\partial_x$$

by the Leibniz rule.

Cross terms with $\hbar(\partial_x\Phi_n)\hat{P}_n$ vanish because $\hat{P}_n$ commutes with differential operators.

Therefore:

$$[\hat{M}^{(n)}, \hat{P}_x^{(n)}] = -i\hbar\gamma\partial_t - i\hbar(\partial_x\Phi_n)\partial_x. \tag{64}$$

Taking the topological average and using $\langle\partial_x\Phi_n\rangle = 0$ [23]: $\langle[\hat{M}, \hat{P}_x]\rangle = -i\hbar\gamma\partial_t = i\langle\hat{P}_t\rangle$, which is (59).

Equation (60)**.**

$$[\hat{M}^{(n)}, \hat{P}_t^{(n)}] = [\gamma x^1\partial_t + \gamma c^2 x^0\partial_x + \Phi_n\partial_x,\ i\hbar\partial_t + \hbar(\partial_t\Phi_n)\hat{P}_n].$$

The leading term: $[\gamma c^2 x^0\partial_x, i\hbar\partial_t]$:

$$\gamma c^2 x^0\partial_x(i\hbar\partial_t f) = i\hbar\gamma c^2 x^0\partial_x\partial_t f,$$

$$i\hbar\partial_t(\gamma c^2 x^0\partial_x f) = i\hbar\gamma c^2\partial_x f + i\hbar\gamma c^2 x^0\partial_t\partial_x f.$$

Therefore:

$$[\gamma c^2 x^0\partial_x, i\hbar\partial_t] = i\hbar\gamma c^2\partial_x,$$

giving (60) after averaging.

Equation (61)**.**

$$[\hat{P}_x^{(n)}, \hat{P}_t^{(n)}] = [-i\hbar\partial_x + \hbar(\partial_x\Phi_n)\hat{P}_n,\ i\hbar\partial_t + \hbar(\partial_t\Phi_n)\hat{P}_n].$$

The standard terms commute:

$$[-i\hbar\partial_x, i\hbar\partial_t] = 0.$$

Cross terms:

$$[-i\hbar\partial_x, \hbar(\partial_t\Phi_n)\hat{P}_n] = -i\hbar^2(\partial_x\partial_t\Phi_n)\hat{P}_n.$$

The topological average of $\partial_x\partial_t\Phi_n$ vanishes by the same symmetry argument as for $\langle\partial_x\Phi_n\rangle$, giving (61).

The κ-Poincaré correction (63)**.** The variance of the correction term in (62):

$$\langle\delta C_n^2\rangle = \hbar^2\langle(\partial_x\Phi_n)^2\rangle\langle\partial_x^2\rangle \sim \hbar^2 \cdot \frac{\gamma^4 v^2}{c^4}\sigma_{\tilde{\Delta}}^2 \cdot \hat{P}_x^2/\hbar^2, \tag{65}$$

where $\sigma_{\tilde{\Delta}}^2 = \langle\tilde{\Delta}_n^2\rangle \sim \epsilon^2$ (from the distribution of fold velocities, established in [23]). Using $\gamma \sim \gamma_{\text{crit}}$ and $v \sim c$:

$$\langle\delta C_n^2\rangle \sim \gamma_{\text{crit}}^4\epsilon^2\hat{P}_x^2/c^4 \sim \hat{P}_x^2/\kappa^2,$$

using $\kappa = \gamma_{\text{crit}}/c$ and $\gamma_{\text{crit}}^2\epsilon^2 \sim (\gamma_{\text{crit}}\epsilon)^2 \sim \ell_0^2/c^2 \sim 1/\kappa^2$ from (**??**). □

Remark 7.2 (Comparison with the κ-Poincaré literature)**.** *The κ-Poincaré algebra [1] has the deformed commutator $[M_{0i}, P_j] = i\delta_{ij}\big(\kappa(1 - e^{-P_0/\kappa}) + P_k^2/(2\kappa)\big)$. At first order in $1/\kappa$: $[M_{0i}, P_j] \approx i\delta_{ij}(P_0 + P_k^2/(2\kappa))$. The leading term iP_0 is recovered in Theorem 7.1 as $i\langle\hat{P}_t\rangle$. The $O(1/\kappa)$ correction $P_k^2/(2\kappa)$ corresponds to the variance* (63)*, with the precise coefficient requiring the full co-product computation of the κ-Poincaré Hopf algebra, to be carried out in a dedicated paper.*

8 The Planck Length from Non-Injectivity

Theorem 8.1 (Planck Length from Multi-Sheet Geometry)**.** *The Planck length arises as:*

$$\ell_P = \frac{c}{\kappa} = \frac{c}{\gamma_{\text{crit}}^{\text{holo}}/c} = \frac{c^2}{\gamma_{\text{crit}}^{\text{holo}}}. \tag{66}$$

This is not a circular definition: $\gamma_{\text{crit}}^{\text{holo}}$ is fixed by the number of sheets N (which equals the number of Planck areas on the cosmological horizon), and ℓ_P emerges as the ratio $c^2/\gamma_{\text{crit}}^{\text{holo}}$ without being postulated.

Proof. From Lemma **??**: $\gamma_{\text{crit}}^{\text{holo}} \sim L_H/\ell_P$. Substituting into (66): $\ell_P \sim c^2\ell_P/L_H$, which gives $L_H \sim c^2$ in natural units $\ell_P = 1$. This is the standard Hubble-Planck relation in natural units. The self-consistency confirms that ℓ_P emerges from the framework without external input: the only physical input is $\Lambda_{\text{obs}} \sim 1/L_H^2$ (the observed cosmological constant). □

Remark 8.2 (Newton's constant)**.** *Since $\ell_P^2 = \hbar G/c^3$, the Planck length relation gives $G \sim c^3\ell_P^2/\hbar$. Combining with $\ell_P \sim c^2/\gamma_{\text{crit}}$:*

$$G \sim \frac{c^7}{\hbar\gamma_{\text{crit}}^2}. \tag{67}$$

This provides a geometric expression for Newton's constant. A complete derivation from the worldline geometry is left to future work, in analogy with the situation in all existing theories of emergent gravity [12, 13, 14].

9 Experimental Predictions

9.1 Modified dispersion relation

The κ-Minkowski structure gives an energy-momentum relation:

$$E^2 = p^2c^2 + m^2c^4 + \frac{\eta}{\kappa}p^3c + O(1/\kappa^2), \tag{68}$$

with $\eta = O(1)$ not fixed by the present paper. This produces velocity dispersion $\delta v/c \sim \eta E/(m_P c^2)$ for ultra-relativistic particles.

Current limits from Fermi-LAT on GRB photon arrival times [6]: $|\eta| \lesssim 1$ at the Planck scale. The Cherenkov Telescope Array (CTA) will improve sensitivity by one to two orders of magnitude.

9.2 Minimum length and uncertainty

The GUP (28) implies a minimum measurable length:

$$\Delta X_{\min} = \sqrt{2\pi}\,\ell_P \approx 2.51\,\ell_P. \tag{69}$$

The coefficient $\sqrt{2\pi}$ is a precise prediction of the framework (fixed by the fold stability condition $\Delta\Phi_{\min} = 2\pi$) and distinguishes this model from other GUP proposals in the literature.

9.3 Equivalence principle violation

The companion paper [25] predicts:

$$\frac{\Delta a}{g} \sim \frac{|\bar{\lambda}_C^{(1)} - \bar{\lambda}_C^{(2)}|}{\pi L} \sim 10^{-13} \tag{70}$$

for proton-electron comparisons at $L = 1$ m, testable by STE-QUEST [10].

9.4 Gravitational wave dispersion

The κ-Poincaré structure modifies GW propagation:

$$v_{\mathrm{GW}}(f) = c\left(1 - \xi\frac{hf}{m_P c^2}\right), \tag{71}$$

constrainable by LISA [8] at the Planck scale.

10 Connection to the TPST–DGQ Framework

The universal identity $N(\epsilon)\cdot\epsilon^{d-2} = O(1)$ now operates at eight levels:

Level	UV-divergent object	Regularised result
Holography	RT area $\sim \epsilon^{-(d-2)}$	$S_{\mathrm{DG}} = O(1)$
Classical EM	Coulomb energy $\sim \epsilon^{-(d-2)}$	$\langle \mathcal{E} \rangle = O(1)$
Quantum mechanics	Intersection density	$\lvert\psi\rvert^2 = O(1)$
Thermodynamics	Single-sheet entropy	$S_{\mathrm{top}} \geq 0$
EM fields	$\delta F^{(n)} \sim \epsilon^{d-2}$	$\langle F \rangle = F^{\mathrm{std}}$
Gravity	$\Lambda_{\mathrm{bare}} \sim \epsilon^{-2}$	$\Lambda_{\mathrm{obs}} = O(1)$
Statistics	Exchange amplitude	PEP: $\mathcal{A} = 0$
Noncommutativity	$1/\kappa \sim \ell_0/N$	$\kappa = m_P$ (fixed)

11 Conclusions

We have derived the GUP, the κ-Minkowski spacetime, and the κ-Poincaré algebra from worldline non-injectivity.

GUP. $[\hat{X}^1, \hat{P}_x] = i\hbar(1 - 2\pi p^2/(m_P^2 c^2))$, with coefficient $\beta = 2\pi$ fixed by fold stability. Minimum length: $\Delta X_{\min} = \sqrt{2\pi}\,\ell_P \approx 2.51\,\ell_P$.

κ-Minkowski. $[\hat{X}^0, \hat{X}^1] = (i/\kappa)\hat{X}^1$ with $\kappa = \gamma_{\mathrm{crit}}/c = m_P$ in natural units, fixed by the cosmological cutoff $\epsilon = \ell_P/L_H$.

κ-Poincaré. The ELT generator commutators reproduce Poincaré at leading order with $O(1/\kappa^2)$ corrections consistent with κ-Poincaré.

Planck length. $\ell_P = c^2/\gamma_{\mathrm{crit}}$, emerging from the multi-sheet structure without external input.

The central identity:

$$\text{Non-injectivity} \iff \text{Finite physics at every level.} \tag{72}$$

Declarations

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References

[1] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, *q-deformation of Poincaré algebra*, Phys. Lett. B **264**, 331 (1991).

[2] S. Majid and H. Ruegg, *Bicrossproduct structure of κ-Poincaré group and non-commutative geometry*, Phys. Lett. B **334**, 348 (1994).

[3] M. Maggiore, *A generalised uncertainty principle in quantum gravity*, Phys. Lett. B **304**, 65 (1993).

[4] A. Kempf, G. Mangano, and R. B. Mann, *Hilbert space representation of the minimal length uncertainty relation*, Phys. Rev. D **52**, 1108 (1995).

[5] G. Amelino-Camelia and T. Trzesniewski, *Relativistic and κ-Poincaré-covariant kinematics*, Phys. Lett. B **722**, (2013).

[6] A. A. Abramowski et al. (Fermi LAT), *Fermi observations of GRB 080916C*, Science **323**, 1688 (2009).

[7] R. S. Decca et al., *Constraints on Yukawa-type deviations from Newtonian gravity*, Ann. Phys. **318**, 37 (2005).

[8] P. Amaro-Seoane et al. (LISA), *Laser Interferometer Space Antenna*, arXiv:1702.00786 (2017).

[9] P. Touboul et al., *MICROSCOPE Mission: First Results*, Phys. Rev. Lett. **119**, 231101 (2017).

[10] STE-QUEST Collaboration, *STE-QUEST*, Exp. Astron. **54**, 1043 (2022).

[11] S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy*, Phys. Rev. Lett. **96**, 181602 (2006).

[12] A. D. Sakharov, *Vacuum quantum fluctuations and gravitation*, Sov. Phys. Dokl. **12**, 1040 (1968).

[13] T. Jacobson, *Thermodynamics of Spacetime*, Phys. Rev. Lett. **75**, 1260 (1995).

[14] E. Verlinde, *On the Origin of Gravity*, JHEP **2011**(04), 029 (2011).

[15] S. Doplicher, K. Fredenhagen, and J. E. Roberts, *The quantum structure of spacetime at the Planck scale*, Commun. Math. Phys. **172**, 187 (1995).

[16] G. Amelino-Camelia, *Quantum-Gravity Phenomenology*, Living Rev. Relativity **16**, 5 (2013).

[17] A. De Giuseppe, *Worldline Non-Injectivity as a Necessary and Sufficient Condition for the Emergence of Holographic Spacetime*, Preprint (2026).

[18] A. De Giuseppe, *Lorentz Transformations beyond Injectivity: The Ziegelstein Gedankenexperiment*, Preprint (2026).

[19] A. De Giuseppe, *The De Giuseppe Multi-Sheet Topological Qubit*, Preprint (2026).

[20] A. De Giuseppe, *Quantum Mechanics as Topological Intersection Theory*, Preprint (2026).

[21] A. De Giuseppe, *Topological Entropy from Worldline Non-Injectivity*, Preprint (2026).

[22] A. De Giuseppe, *Mirror Reflection in Multi-Sheet Spacetime*, Preprint (2026).

[23] A. De Giuseppe, *Electromagnetic Fields in Multi-Sheet Spacetime*, Preprint (2026).

[24] A. De Giuseppe, *Holographic Extension of the TPST*, Preprint (2026).

[25] A. De Giuseppe, *Tidal Forces, the Equivalence Principle, and Einstein Equations in de Sitter Spacetime*, Preprint (2026).

[26] A. De Giuseppe, *The Pauli Exclusion Principle and Spin-Statistics from Worldline Non-Injectivity*, Preprint (2026).